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On the estimation of Weibull's parameters in brittle materials

M. LEON, P. KITTL
Instituto de Investigaciones y Ensayes de Materiales (IDIEM), Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Casilla 1420, Santiago, Chile

The parameters in Weibull's specific risk function are estimated using several methods, both in compression and in bending. To this end, several formulae are employed for different values of the limiting stress σ_L. The parameters are estimated by minimizing χ^2 in order to compare the capacity of the different formulae for approximating these parameters. In the case of bending, the error committed when obtaining the parameters from a uniform-stress-field Weibull function is of the order of 100%.

1. Introduction

From the time when the Weibull theory [1] was established, one of the first works to estimate Weibull's parameters was that of Jayatilaka and Trustrum [2], but in that work the bending was approximated using a uniaxial constant stress. This procedure has already been discussed by Kittl and Günther [3], and some differences in the estimation of the parameters are pointed out below.

In a recent work, Glandus and Boch [4] have used this approximation in order to determine the uncertainty in Weibull's modulus as a function of the number of samples tested.

The present work has been done in order to show the differences in the estimation of Weibull's parameters in bending using, on the one hand, an exact expression for the cumulative probability of fracture, previously obtained by Kittl [5], and, on the other, the approximate formula calculated on the basis of a uniaxial constant stress field.

2. On the estimation of Weibull's parameters in a uniaxial compression or traction stress field

In the general theory of fracture statistics [6] for a uniaxial variable stress field $\sigma(\bar{r})$, the cumulative probability of fracture $F(\sigma)$ is

$$F(\sigma) = 1 - \exp\left(-\frac{1}{V_0}\int_V \phi[\sigma(\bar{r})]\,dV\right) \quad \bar{r} \in V \tag{1}$$

where $\bar{r}$ is the position vector, V is the body volume, V_0 is the unit volume and $\phi(\sigma)$ is Weibull's specific risk function. Usually this function $\phi(\sigma)$ is

$$\phi(\sigma) = \begin{cases} [(\sigma - \sigma_L)/\sigma_0]^m & \sigma \geqslant \sigma_L \\ 0 & \sigma < \sigma_L \end{cases} \tag{2}$$

where m and σ_0 are parameters and σ_L is the limiting stress below which there is no fracture. When $\sigma(\bar{r})$ is constant, Equation 1 is transformed into

$$F(\sigma) = 1 - \exp\left(-\frac{V}{V_0}\phi(\sigma)\right) \tag{3}$$

which in turn is easily transformed into

$$\xi(\sigma) = \ln\left(\frac{1}{1 - F(\sigma)}\right) = \frac{V}{V_0}\phi(\sigma) \tag{4}$$

In order to determine the parameters in Weibull's specific risk function, many works have been published [2, 7–9], with diverse methods for estimating these parameters, such

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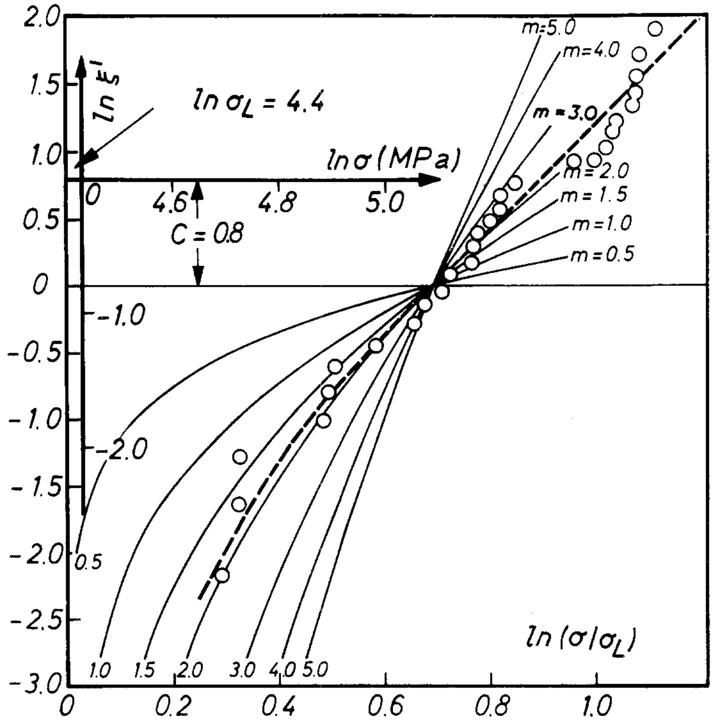

Figure 1 Weibull function's nomogram for a uniform stress field in the case of a traction or compression test, where $\ln \xi' = m \ln (\sigma/\sigma_L - 1)$ is plotted as a function of $\ln (\sigma/\sigma_L)$ for several m. The circles correspond to 30 glass cylinders broken by compression. In order to obtain m, σ_L and σ_0 from the nomogram, the points are drawn on transparent paper, and moved in order to be superimposed on the nomogram curve that best fits the experimental points; this gives m. The constants σ_L and $C = C(\sigma_0, \sigma_L, m)$ are obtained from the distances between the reference axis of the experimental points on the transparent paper and the nomogram axis.

as the old and well known least-squares method, more special procedures such as the method of moments and the minimum chi-square method, and recently the laborious maximum-likelihood method.

From the practical viewpoint, it is important to get an approximation of the parameters by means of a method that is speedy and less expensive. The formulation of such a convenient method is one of the problems solved in the present paper. Thus, for estimating Weibull's parameters in a uniaxial compression or traction stress field, a graphical method is developed herein. This method consists of obtaining the said parameters by comparison with a non-dimensional nomogram.

To find the non-dimensional curves of Weibull's plot, it is necessary to proceed as follows. From Equations 4 and 2 we have

$$\xi(\sigma) = \frac{V}{V_0}\left(\frac{\sigma - \sigma_L}{\sigma_0}\right)^m \qquad (5)$$

Rearranging this and taking logarithms we get

$$\ln [\xi(\sigma)] = \ln\left[\frac{V}{V_0}\left(\frac{\sigma_L}{\sigma_0}\right)^m\right] + m \ln\left(\frac{\sigma}{\sigma_L} - 1\right) \qquad (6)$$

Now if we plot $m \ln (\sigma/\sigma_L - 1)$ against $\ln (\sigma/\sigma_L)$, for various values of Weibull's modulus m, as shown in Fig. 1 we obtain a non-dimensional graph, called a nomogram for constant stress field. The parameters m, σ_0 and σ_L in a particular case are obtained by drawing the Weibull plot on transparent paper, and moving it in order that the experimental points can be fitted to a nomogram curve to determine m. Then $\ln \xi$ is on the $m \ln (\sigma/\sigma_L - 1)$ axis as a function of $\ln \sigma$. Consequently, from the $\ln \xi$ axis is obtained

$$C = \ln\left[\frac{V}{V_0}\left(\frac{\sigma_L}{\sigma_0}\right)^m\right]$$

and from the $\ln \sigma$ axis is obtained $\ln \sigma_L$.

The results obtained using this method were compared with more sophisticated methods using a set of data given in Kittl *et al.* [7], consisting of a batch of 30 samples of commercial glass rods 0.007 m in diameter and 0.014 m high; hence the volume of a rod is 0.539 cm^3. The faces of the cylinders were polished to get a uniform compression stress and then the cylinders were broken using a Monsanto Manual Machine. As Kittl and Aldunate [6] demonstrated that their observed compression failure stress in compacted cement cylinders was fitted better by a log-normal distribution than by a Weibull distribution data for, glass cylinders were used

TABLE I Statistical parameters of Weibull's distribution for a batch of 30 glass rods 0.007 m in diameter and 0.014 m high (units in MPa).

Parameters	Method			
	Nomogram	Least squares	Moments	Maximum likelihood
m	2.0	2.2	2.2	2.2
σ_0 ($\times 10^{-2}$)	89.1	16.9	15.7	15.6
σ_L	81.2	83.3	89.8	90.5
$\chi^2 < \chi^2_{0.95,1}$	3.82	2.93	2.83	2.87

$\chi^2_{0.95,1} = 3.84$; degrees of freedom, $\nu = 1$, with five classes.

because the Weibull distribution is better than any other in this case [7].

The final result is presented in Table I, which shows that the parameters estimated by means of the nomogram have a small uncertainty in comparison with the parameters obtained using sophisticated methods. This uncertainty is acceptable inasmuch as the nomogram method is the most speedy and least expensive of all the diverse methods mentioned here.

3. On the estimation of Weibull's parameters under flexure

The general theory of fracture statistics predicts the correct expression by means of the cumulative probability of failure developed by Kittl [5] for tests applying three-point bending.

For the case of a rectangular bar of length L, width b and height h, subject to a flexural test applying a fracture load P at the centre, the cumulative probability of failure is

$$F(\sigma) = 1 - \exp\left[-\frac{bhL}{2(m+1)V_0}\left(\frac{\sigma_L}{\sigma_0}\right)^m \frac{1}{\sigma/\sigma_L} \times \int_1^{\sigma/\sigma_L} \frac{(\eta-1)^{m+1}}{\eta}\,d\eta\right] \quad (7)$$

where $\sigma = \frac{3}{2}PL/bh^2$ is the maximum stress in the body at fracture under load P. Obviously Equation 7 is difficult to use to find the parameters m, σ_0 and σ_L. For this reason, authors sometimes use the form wherein $\sigma_L = 0$, namely

$$F(\sigma) = 1 - \exp\left[-\frac{bhL}{2(m+1)^2V_0}\left(\frac{\sigma}{\sigma_0}\right)^m\right] \quad (8)$$

whose mathematical treatment is easy, or the cumulative probability of failure for a constant uniaxial compression or traction stress field

$$F(\sigma) = 1 - \exp\left[-\frac{bhL}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^m\right] \quad (9)$$

Equations 8 and 9 exhibit analogous form, with the same parameter m but with different parameters σ_0. This is easily seen by linearizing the equations as follows:

$$\ln\left[\ln\left(\frac{1}{1-F(\sigma)}\right)\right] = \ln\left(\frac{bhL}{2(m+1)^2V_0\sigma_0^m}\right) + m\ln\sigma \quad (10)$$

$$\ln\left[\ln\left(\frac{1}{1-F(\sigma)}\right)\right] = \ln\left(\frac{bhL}{V_0\sigma_0^m}\right) + m\ln\sigma \quad (11)$$

The first method developed is the non-dimensional graph denominated nomogram for flexure, which is obtained by rearranging Equation 7; then it is possible to get the following non-dimensional form:

$$\ln\xi(\sigma) = \ln\left[\frac{bhL}{2(m+1)^2V_0}\left(\frac{\sigma_L}{\sigma_0}\right)^m\right] + m\ln\left(\frac{\sigma}{\sigma_L}\right) + \ln\left(\frac{m+1}{(\sigma/\sigma_L)^{m+1}}\int_1^{\sigma/\sigma_L}\frac{(\eta-1)^{m+1}}{\eta}\,d\eta\right) \quad (12)$$

Then plotting

$$m\ln(\sigma/\sigma_L) + \ln\left(\frac{m+1}{(\sigma/\sigma_L)^{m+1}}\int_1^{\sigma/\sigma_L}\frac{(\eta-1)^{m+1}}{\eta}\,d\eta\right)$$

against $\ln(\sigma/\sigma_L)$, for various values of parameter m, we obtain the nomogram for flexure shown in Fig. 2. The parameters m, σ_0 and σ_L for the cumulative probability of failure under flexure are obtained by comparison between the

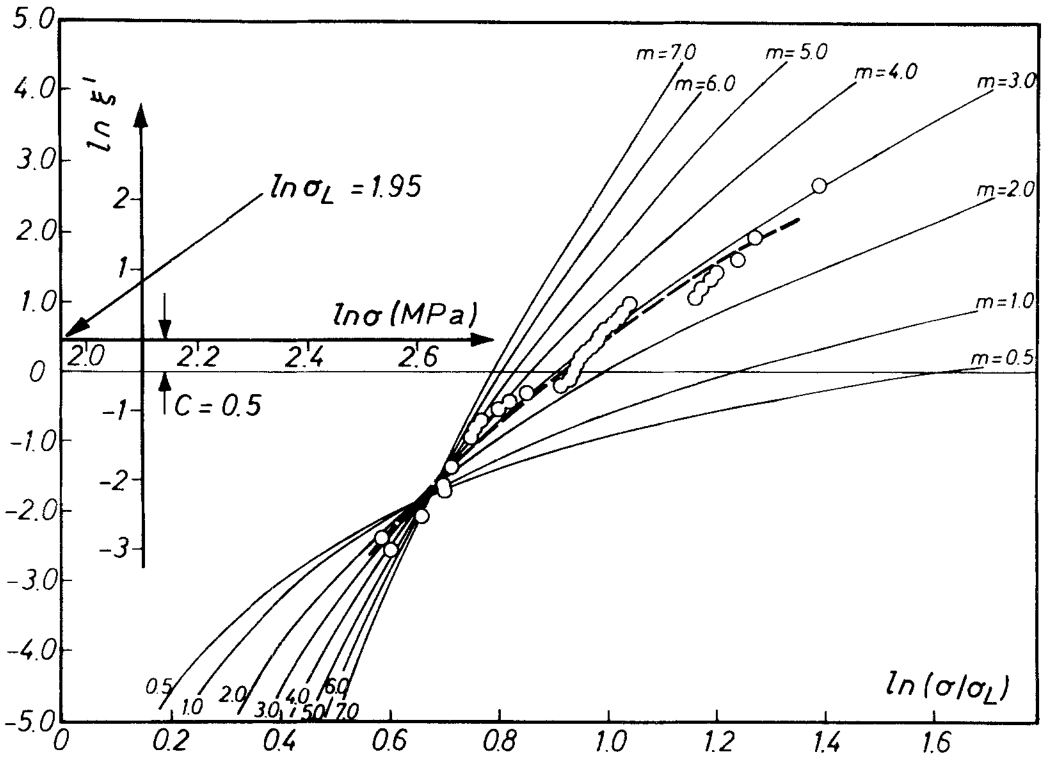

Figure 2 Weibull function's nomogram for a variable stress field in the case of a rectangular beam subjected to a flexural test of three-point bending, where

$$\ln \xi' = m \ln (\sigma/\sigma_L) + \ln \left(\frac{m+1}{(\sigma/\sigma_L)^{m+1}} \int_1^{\sigma/\sigma_L} \frac{(\eta - 1)^{m+1}}{\eta} \, d\eta \right)$$

is plotted as a function of $\ln (\sigma/\sigma_L)$. The points within the circles come from 32 compacted cement paste bars broken by flexion. Here m, σ_L and $C = C(\sigma_0, \sigma_L, m)$ are obtained in the same way as in Fig. 1.

Weibull plot and the nomogram in the same way as explained above, where from the $\ln \xi$ axis is obtained

$$C = \ln \left[\frac{bhL}{2(m+1)^2 V_0} \left(\frac{\sigma_L}{\sigma_0} \right)^m \right]$$

and from the $\ln \sigma$ axis is obtained $\ln \sigma_L$.

The second method developed is the minimum chi-square method [10, 11]. This procedure is similar to the maximum-likelihood method; however this latter procedure is applicable to more general problems. The minimum chi-squares are asymptotically efficient and squared error-consistent estimators under quite general conditions. The χ^2 is

$$\chi^2 = \sum_{i=1}^{r} \frac{[k_i - nP_i(m, \sigma_0, \sigma_L; \sigma_i)]^2}{nP_i(m, \sigma_0, \sigma_L; \sigma_i)} \quad (13)$$

where the population is classified into r classes each comprising k_i elements, n is the number of trials $\Sigma_{i=1}^{r} k_i = n$ and P_i is the probability of failure in the classes. Then the minimum condition for chi-squared is

$$-\frac{1}{2} \frac{\partial \chi^2}{\partial \alpha_j} = \sum_{i=1}^{r} \left(\frac{k_i - nP_i}{P_i} + \frac{(k_i - nP_i)^2}{2nP_i} \right) \times \frac{\partial P_i}{\partial \alpha_j} = 0 \quad (14)$$

where $\alpha_j = (m, \sigma_0, \sigma_L)$ and $j = 1, 2, 3$. Therefore the estimators of minimum chi-square are obtained by solving the nonlinear system of Equation 14. For large samples the second term appearing within the parentheses is neglected; in this case, in general it is easier to solve the following system

$$\sum_{i=1}^{r} \left(\frac{k_i - nP_i}{P_i} \right) \frac{\partial P_i}{\partial \alpha_j} = 0 \qquad j = 1, 2, 3 \quad (15)$$

Owing to the complicated nature of the cumulative probability of failure under flexure, it is very difficult to solve the systems of Equations 14 or 15. Hence it is necessary to program an algorithm with back-tracking to reach the minimum chi-square using the general Equation 13.

TABLE II Statistical parameters of Weibull's distribution for a batch of 32 samples of compacted cement paste, broken under flexure (units in MPa)

Parameters	Flexure				Traction	
	$\sigma_L \neq 0$		$\sigma_L = 0$		$\sigma_L \neq 0$	$\sigma_L = 0$
	Nomogram	χ^2 minimum	Least squares	χ^2 minimum	Least squares	Least squares
m	3.00	3.09	6.43	6.11	1.836	6.43
σ_0	6.00×10^{-2}	6.38×10^{-2}	1.51	1.35	1.23×10^{-2}	3.13
σ_L	6.96	7.03	–	–	11.67	–
$\chi^2 < \chi^2_{0.95,1}$	3.14	1.67	2.90	2.85	2.92	2.89

$\chi^2_{0.95,1} = 3.84$; degrees of freedom, $\nu = 1$, with five classes.

The back-tracking algorithm is very efficient for finding the minimum of the chi-square function; this algorithm is more useful in games and simulations that allow one to find the solution of a problem through a discrete number of steps. One of the problems regarding this method is the case of a small sample, for which it is not applicable. Another difficulty is the choice of adequate classes having at least five elements each.

In order to compare the efficiency of the different methods, data supplied by Kittl and Günther [12] were used, consisting of a batch of 32 samples of compacted cement paste, 0.0635 m long, 0.023 m wide and 0.005 77 m high. These rectangular bars were broken applying the three-point bending test. The results are shown in Table II; moreover, the parameters used for the cumulative probability of failure for a constant stress field (traction or compression) and for the cumulative probability of failure under flexure with $\sigma_L = 0$ were computed. The chi-square test was also done to compare the several methods and distributions employed for describing the behaviour of materials.

Table II evidently shows the necessity of using Equation 7 for the cumulative probability of fracture under flexure, with three parameters, to represent the behaviour of Weibull material under bending. Besides, when using a formula for the cumulative probability of fracture for constant stress field to define the behaviour of a Weibull material under flexure (variable stress field), the error committed in the estimation of parameters m, σ_0 and σ_L amounts from 60% to 700%.

Hence, in general, we must conclude that in a variable stress field it is not possible to obtain Weibull's parameters with an acceptable approximation if the stress field is approximated using a constant field.

If Weibull statistics are used, it is assumed that the material follows their laws, and consequently the correct expressions must be used for the stress field, as has already been pointed out by Kittl and Günther [3].

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References
1. W. WEIBULL, *Ing. Vetenskaps Akad. Handl.* **151** (1939) 1.
2. A. de S. JAYATILAKA and K. TRUSTRUM, *J. Mater. Sci.* **14** (1979) 1080.
3. P. KITTL and O. GÜNTHER, *ibid.* **17** (1982) 922.
4. J. C. GLANDUS and P. BOCH, *J. Mater. Sci. Lett.* **3** (1984) 74.
5. P. KITTL, *Res. Mech.* **1** (1980) 161.
6. P. KITTL and R. ALDUNATE, *J. Mater. Sci.* **18** (1983) 2947.
7. P. KITTL, M. LEON and G. M. CAMILO, "Advances in Fracture Research", Vol. 4 (Pergamon Press, Oxford, 1985).
8. M. LEON and P. KITTL, *Latin Am. J. Met. Mater.* in press.
9. W. TRADINIK, K. KROMP and R. E. PABST, *Materialprüf.* **23** (1981) 42.
10. A. MOOD and F. A. GRAYBILL, "Introduction to the Theory of Statistics" (McGraw Hill, New York, 1962) pp. 308–11.
11. H. CRAMER, "Métodos Matemáticos de Estadística" (Aguilar SA de Ediciones, Madrid, 1970) pp. 477–518.
12. P. KITTL and O. GÜNTHER, *Res. Mechanica Lett.* **1** (1981) 145.

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